REFERENCES

1. GUZ' A.N. and GOLOVCHAN V.T., Difraction of elastic waves in multicoupled bodies. Kiev: Nauka. Dumka, 1972.
2. BABESHKO V.A., SELEZNEV M.G., SELEZNEVA T.N. and SOKOLOV V.P., One method for analyzing steady oscillations of an elastic half-space containing a cavity. PMM, Vol.47, No.1, 1983.
3. MORSE F. and ESCHBACH G., Methods of theoretical physics. Vol.2. Moscow: Izd-vo inostr. lit., 1960.
4. NOVATSKII V., Theory of elasticity. Moscow: Mir, 1975.
5. FRANK F. and MISES R., Differential and integral equations of mathematical physics. Moscow-Leningrad: Gostekhizdat, 1973.
6. IVANOV E.A., Diffraction of electromagnetic waves by two bodies. Minsk: Nauka i tekhnika, 1968.
7. ALEKSEYEVA L.A., Oscillations of an elastic half-plane when influenced by a stationary source of cylindrical waves. Izv. AN KazSSR. Ser. fiz.-matem., No.5, 1983.
8. Handbook on special functions with formulas, graphs and mathematical tables. Moscow: Nauka, 1979.
9. KANTOROVICH L.V. and KRYLOV V.I., Approximate methods of higher analysis. Leningrad: Fizmatgiz, 1962.

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REFRACTION OF PLANE-POLARIZED WAVES AT THE BOUNDARY OF AN ELASTIC AND ELASTOPLASTIC HALF-SPACE*

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Selfsimilar solutions of dynamic equations for antiplane deformation in an ideal elasto plastic medium are considered. A solution is constructed of the problem of the refraction of plane-polarized plane waves of an arbitrary profile which penetrate from the elastic to the elastoplastic half-space.

Selfsimilar solutions were investigated earlier /1-4/ when the rates of displacements and stresses depend only on the ratio of the coordinates. The selfsimilar problem of the refraction of a plane elastic wave into an elastoplastic half-space with boundary conditions like those of Coulomb's law of dry friction, and conditions guaranteeing full contact at the boundary of separation, were analysed in /5, 6/.

1. Consider the dynamic problem of the theory of complex displacement in an ideal elastoplastic medium. In a rectangular Cartesian system of coordinates x_i the vector of the rate of displacement w is directed along x_3 axis and depends only on x_1, x_2 and the time t.

All the components of stress vanish, apart from $\tau_1 = \sigma_{13}(x_1, x_2, t)$, $\tau_2 = \sigma_{23}(x_1, x_2, t)$. The equations of motion in this case have the form

$$\frac{\partial \tau_1}{\partial x_1} + \frac{\partial \tau_2}{\partial x_2} - \rho \frac{\partial w}{\partial t} = 0 \,. \tag{1.1}$$

The full deformations are the sum of the elastic and the plastic part, and the elastic deformations are connected with the stresses by Hooke's law

$$\gamma_1 = \gamma_1^e + \gamma_1^p, \quad \gamma_2 = \gamma_2^e + \gamma_2^p; \quad \tau_1 = 2\mu\gamma_1^e, \quad \tau_2 = 2\mu\gamma_2^e \,. \tag{1.2}$$

In the plastic domain, the stresses satisfy the condition of plasticity, and the rates of the plastic deformations are determined from the associated flow rule

$$\tau_1^2 + \tau_2^2 = k^2; \; \gamma_1^{\cdot p} = \psi\tau_1, \; \gamma_2^{\cdot p} = \psi\tau_2 \,. \tag{1.3}$$

The total rates of deformation are expressed in terms of the displacements by

$$\gamma_1^{\cdot} = \frac{1}{2}\frac{\partial w}{\partial x_1}, \quad \gamma_2^{\cdot} = \frac{1}{2}\frac{\partial w}{\partial x_2} \,. \tag{1.4}$$

Differentiating relations (1.2) with respect to time and eliminating the values of the

*Prikl.Matem.Mekhan., 49, 2, 307-315, 1985 rates of deformations, we obtain

$$\frac{\partial \tau_1}{\partial t} = \mu \frac{\partial w}{\partial x_1} - 2\mu\psi\tau_1, \quad \frac{\partial \tau_2}{\partial t} = \mu \frac{\partial w}{\partial x_2} - 2\mu\psi\tau_2 . \tag{1.5}$$

The first Eq.(1.3) will hold identically if we assume

$$\tau_1 = k \sin\theta, \ \tau_2 = k\cos\theta . \tag{1.6}$$

Substituting these values into Eqs.(1.1) and (1.5), after eliminating ψ we obtain a set of equations for determining θ and w. Consider the selfsimilar solution of this system, when the functions w and θ depend only on $x = x_1 - ct$, $y = x_2$. This system then takes the form

$$k\left(\cos\theta \frac{\partial\theta}{\partial x} - \sin\theta \frac{\partial \theta}{\partial y}\right) + \rho c \frac{\partial w}{\partial x} = 0 \tag{1.7}$$
$$\mu\left(\cos\theta \frac{\partial w}{\partial x} - \sin\theta \frac{\partial w}{\partial y}\right) + kc \frac{\partial \theta}{\partial x} = 0 .$$

The set of Eqs.(1.7) is of the hyperbolic type. Its characteristics and relations along the characteristics have the form

$$dy\,(M - \cos\theta) = \sin\theta\, dx, \ k\theta - \rho a w = \text{const} \tag{1.8}$$

$$dy\,(M + \cos\theta) = -\sin\theta\, dx, \ k\theta + \rho a w = \text{const} \tag{1.9}$$

where $a = \sqrt{\mu/\rho}$ is the velocity of longitudinal elastic waves, and $M = c/a$ is the Mach number.

In the elastic domain and unloading zone the plastic deformations equal zero; then from Eqs.(1.1)-(1.4) we obtain

$$c\tau_1 + \mu w = f(y) \tag{1.10}$$

$$c\frac{\partial \tau_2}{\partial x} + \mu \frac{\partial w}{\partial y} = 0, \quad c\frac{\partial \tau_2}{\partial y} + (\rho c^2 - \mu)\frac{\partial w}{\partial x} = 0 . \tag{1.11}$$

The set of Eqs.(1.11) when $c > \sqrt{\mu/\rho}$ is of the hyperbolic type. The characteristics and relations along the characteristics have the form

$$x - \sqrt{M^2 - 1}\, y = \text{const}, \ \mu\sqrt{M^2-1}\, w - c\tau_2 = \text{const} \tag{1.12}$$

$$x + \sqrt{M^2 - 1}\, y = \text{const}, \ \mu\sqrt{M^2-1}\, w - c\tau_2 = \text{const} . \tag{1.13}$$

The solution of boundary value problems of wave dynamics for elastoplastic media is reduced to a determination of the solutions of Eqs.(1.11) in the elastic domain and of the solutions of Eqs.(1.7) in a plastic domain and to finding the boundary of separation between them from the conditions of continuity of the stresses, displacements and plastic deformations.

Note that we need confine ourselves only to those solutions of the Eqs.(1.7) for which the energy dissipation $D = \gamma_1^{\cdot p}\tau_1 + \gamma_2^{\cdot p}\tau_2 \geqslant 0$ at each point. Using relations (1.2), (1.4) and (1.6), the condition for the energy dissipation to be positive can be written in the form

$$D = \frac{1}{2} k\left(\frac{\partial w}{\partial x}\sin\theta + \frac{\partial w}{\partial y}\cos\theta\right) \geqslant 0 \tag{1.14}$$

2. We shall use the relations obtained to investigate refraction of the waves which pass from the elastic half-space with parameters μ_1 ρ_1, $a_1 = \sqrt{\mu_1/\rho_1}$ to the elastoplastic half-space with parameters $\mu_2, \rho_2, a_2 = \sqrt{\mu_2/\rho_2}$. Suppose the plane wave OA (Fig.1) falls on the surface of separation $y = 0$. The equation of the incident wavefront at any instant of time has the form $y\cos\varphi_1 + x\sin\varphi_1 = \text{const}$. Behind the incident wavefront in the elastic half-space the following relations hold:

$$\tau_1 = \tau_1(\omega_1), \ \tau_2 = \tau_2(\omega_1), \ w = w_1(\Omega_1); \ \Omega_1 = -y\cos\varphi_1 - x\sin\varphi_1 . \tag{2.1}$$

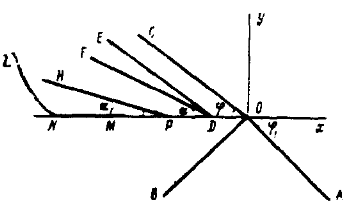

Fig.1

The equation of the reflected wavefront OB has the form $y\cos\varphi_1 - x\sin\varphi_1 = \text{const}$. Behind the reflected wavefront in the elastic half-space a solution is obtained by combining the solution (2.1) with the solution for the reflected wave

$$\tau_1 = \tau_1(\Omega_2), \ \tau_2 = \tau_2(\Omega_2), \ w = w_2(\Omega_2); \ \Omega_2 = y\cos\varphi_1 - x\sin\varphi_1 . \tag{2.2}$$

From Eqs.(1.10) and (1.11) it follows that

$$\tau_1(\Omega_1) = -c^{-1}\mu_1 w_1(\Omega_1), \ \tau_2(\Omega_1) = -c^{-1}\mu_1 \operatorname{ctg}\varphi_1 w_1(\Omega_1) \tag{2.3}$$
$$\tau_1(\Omega_2) = -c^{-1}\mu_1 w_2(\Omega_2), \ \tau_2(\Omega_2) = c^{-1}\mu_1 \operatorname{ctg}\varphi_1 w_2(\Omega_2) .$$

At the boundary of separation $y = 0$ the stresses τ_2 and rate of displacements w are continuous, whence

$$w(x) = w_1(-x\sin\varphi_1) + w_2(-x\sin\varphi_1)$$
$$\tau_2(x) = c^{-1}\mu_1 \operatorname{ctg}\varphi_1 (w_2(-x\sin\varphi_1) - w_1(-x\sin\varphi_1)) \tag{2.4}$$

where $w(x)$ is the rate of the displacements, and $\tau_2(x)$ is the stress at the boundary of separation in the elastoplastic half-space.

Eliminating the function $w_2(-x\sin\varphi_1)$ from relations (2.4), we obtain the boundary condition for the elastoplastic half-space

$$2w_1(-x\sin\varphi_1) = w(x) - c\mu_1^{-1}\operatorname{tg}\varphi_1\tau_2(x). \tag{2.5}$$

It is assumed below that the function $w_1(\Omega_1)$ is known, i.e. the profile of the incident wave is given.

Consider the refracted wave in the elastoplastic half-space. In front of the refracted wavefront OC, whose equation has the form $y\cos\varphi + x\sin\varphi = 0$, the material is assumed to be at rest: $w = \tau_1 = \tau_2 = 0$, i.e. in the neighbourhood of the line OC the material will be in an elastic state.

Since when $x = \infty$ we have $\tau_1 = 0$ and $w = 0$, then from (1.10) it follows that

$$\tau_1 = -c^{-1}\mu_2 w. \tag{2.6}$$

Relations (1.12) and (1.13) take the form

$$x - \sqrt{M^2 - 1}\, y = \text{const}, \ \mu_2\sqrt{M^2-1}\, w + c\tau_2 = 0;\ M = c\, a_2 \tag{2.7}$$

$$x + \sqrt{M^2 - 1}\, y = \text{const}, \ \mu_2\sqrt{M^2-1}\, w - c\tau_2 = \text{const}. \tag{2.8}$$

On the right-hand side of Eq.(2.7) the constant is put equal to zero, since on the line OC we have $\tau_2 = 0$, $w = 0$, therefore (2.7) can be considered as an integral of the equation of motion of the elastic medium. Using (2.7), we can represent the condition at the edge (2.5) in the form

$$2w_1(-x\sin\varphi_1) = w(x)\left(1 + \frac{\mu_2\operatorname{tg}\varphi_1}{\mu_1\operatorname{tg}\varphi}\right). \tag{2.9}$$

It follows from the integrals (2.7) and (2.8) that the rates of displacements w and the stresses τ_1, τ_2 remain constant along the characteristics (2.8). Therefore the yield point will be reached immediately on all the characteristics, if it is reached at least at one point.

It follows from the condition for reaching the yeild point

$$\tau_1^2 + \tau_2^2 = a_2^{-2}\mu_2^2 w^2 = \mu_2\rho_2 w^2 = k^2 \tag{2.10}$$

and condition (2.9), that the material will remain elastic between the characteristics OC and DE (Fig.1), until the following equality is achieved at some point D of the boundary:

$$2\,|w_1^*(-x\sin\varphi_1)| = \frac{k}{\sqrt{\mu_2\rho_2}}\left(1 + \frac{\mu_2\operatorname{tg}\varphi_1}{\mu_1\operatorname{tg}\varphi}\right). \tag{2.11}$$

On the line DE

$$w = \frac{k}{\sqrt{\mu_2\rho_2}}, \quad \tau_1 = -k\sin\varphi, \quad \tau_2 = -k\cos\varphi. \tag{2.12}$$

To the left of the line DE the material is in a plastic state, where (1.8) and (1.9) occur.

Since on the line DE we have $\theta = \pi + \varphi$, the characteristics of (1.9) intersect the line DE and the following relations hold on them:

$$dy(M + \cos\theta) = -\sin\theta\, dx, \ k\theta - \rho_2 a_2 w = k(1 + \pi + \varphi). \tag{2.13}$$

Since on the right-hand side of the second relation (2.13) the constant is one and the same for all the characteristics, this equation should be considered as an integral of the equations of motion in the plastic domain. From the integral (2.13) and relations (1.8) we find that along each of the characteristics of the other family, w and θ do not change, whence it follows that the characteristics (1.8) are rectilinear. Thus, we have

$$y(M - \cos\theta) - x\sin\theta = \text{const}, \ k\theta - \rho_2 a_2 w = \text{const}. \tag{2.14}$$

The characteristics (2.14) intersect the line DE and incline towards the x axis under the angle α, for which

$$\operatorname{tg}\alpha = -\sin\varphi/(M + \cos\varphi) \leqslant \operatorname{tg}\varphi .$$

We therefore have a Cauchy problem for the equations of motion in a plastic domain on the line DE, by solving which we determine θ and w between the characteristics in the elastic domain DE and the characteristics in the plastic domain DF, where

$$w = k/\sqrt{\mu_2\rho_2},\ \theta = \pi + \varphi . \tag{2.15}$$

From the boundary condition (2.5) and the integral (2.13) after eliminating w, we obtain

$$2\mu_1 (ck)^{-1} \operatorname{ctg}\varphi_1 w_1 (-x\sin\varphi_1) = \Delta (1 + \pi + \varphi - \theta) - \cos\theta \tag{2.16}$$

$$\Delta = \frac{\mu_1 \sin\varphi}{\mu_2 \sin\varphi_1}\cos\varphi_1 .$$

If $\Delta \geqslant 1$, then for any value of the left-hand side of Eq.(2.16) the latter has a unique solution. If $\Delta < 1$, Eq.(2.16) can have three or more roots relative to θ. On the line DE only the root $\theta_1 = \pi + \varphi$ reduces to a continuous conjugation of the solutions in the elastic and plastic domain. Therefore in the plastic domain we should choose the root which - when x approaches x_D - approaches $\pi + \varphi$.

Suppose this is the root $\theta = \theta_1$. Then from Eq.(2.13) at the edge $y = 0$ we obtain

$$w = \frac{k}{\rho_2 a_2}(1 + \pi + \varphi - \theta_1).$$

The values θ_1 and w remain constant along the line

$$y (M - \cos\theta_1) - (x - x_p(\theta_1))\sin\theta_1 = 0 \tag{2.17}$$

where x_p is the coordinate of the boundary point, at which $\theta = \theta_1$. The angle of inclination to the x axis of this characteristic is connected with θ_1 by the relation

$$\operatorname{tg}\alpha_1 = -\sin\theta_1/(M - \cos\theta_1). \tag{2.18}$$

For the solution to occur, the angle α_1 must increase and the dissipation of energy D in the plastic domain should be positive for the motion of the point H along the x axis.

It follows from (2.18) that $\partial\alpha_1/\partial x > 0$, if

$$(1 - M\cos\theta_1)\,\partial\theta_1/\partial x \geqslant 0 . \tag{2.19}$$

Since θ_1 satisfies Eq.(2.15), then when the argument increases - as long as function $w(\Omega_1)$ increases, $d\theta_1/dx > 0$, and inequality (2.19) will also hold if

$$1 - M\cos\theta_1 \geqslant 0 . \tag{2.20}$$

At the point D we have $\theta_1 = \pi + \varphi$, i.e. the inequality (2.20) clearly holds. Since to the left of the point D we have $d\theta_1/dx > 0$, then θ_1 decreases as w_1 increases. Decreasing, θ_1 can attain the value π when the characteristic in the plastic domain becomes parallel to the x axis. This is possible if the amplitude of the incident wave attains the value

$$w_1^\circ = \frac{k}{2\sqrt{\mu_1\rho_1}}\left(\frac{\mu_1\sin\varphi}{\mu_2\sin\varphi_1}(1+\varphi) + \frac{1}{\cos\varphi_1}\right). \tag{2.21}$$

Suppose this value is attained at the point M and henceforth, as the argument increases, the function $w_1(\omega_1)$ continues to increase, then the line MN is a characteristic and on it $\theta_1 = \pi$, and $\tau_1 = 0$, $\tau_2 = -k$, $w = k(1+\varphi)(\rho_2 a_2)^{-1}$. The solution in the upper half-space is determined by the boundary condition (2.16) on the line ON. The line MN is a stationary line of the discontinuity on which the rates of the displacements undergo a discontinuity, and it follows from the dynamic conditions of compatibility on the surface of the strong discontinuity that the quantity τ_2 is continuous on the line MN. From (2.4) we obtain the intensity of the reflected wave

$$w_2(-x\sin\varphi_1) = w_1(-x\sin\varphi_1) - \frac{k}{\sqrt{\mu_1\rho_1}\cos\varphi_1} . \tag{2.22}$$

Consider the energy dissipation in the plastic domain. From (1.14) and (2.13) we obtain

$$D = -\frac{k^2}{2\rho_2 a_2}\left(\frac{\partial\theta}{\partial x}\sin\theta + \frac{\partial\theta}{\partial y}\cos\theta\right). \tag{2.23}$$

From Eq.(2.17) follows

$$z(\theta)\frac{\partial\theta}{\partial x} - \sin\theta = 0, \quad z(\theta)\frac{\partial\theta}{\partial y} + M - \cos\theta = 0 \tag{2.24}$$

$$\left(z(\theta) = \left(y + \frac{dx_p(\theta)}{d\theta}\right)\sin\theta - (x - x_p(\theta)\cos\theta)\right).$$

From (2.23) and (2.24) we obtain the condition for the energy dissipation $z(\theta)(1-M\cos\theta)^{-1}\leqslant 0$ to be positive, which, using (2.20) and (2.17), we can transform to the form

$$\left(y(1-M\cos\theta)+\frac{dx_p(\theta)}{d\theta}(\sin\theta)^2\right)(\sin\theta)^{-1}\leqslant 0. \tag{2.25}$$

In the plastic domain $\pi\leqslant\theta\leqslant\pi+\varphi$, $y\geqslant 0$, therefore the inequaltiy (2.25) occurs at any point of the plastic domain when $\partial\vartheta/\partial x\geqslant 0$. Thus, $D>0$ at all points of the plastic domain. When $w_1>w_1^0$ the energy dissipation D in the plastic domain is also positive, but we should especially consider the dissipation when $y=0$ in the zone of slippage. On the stationary line of the discontinuity of the rates of displacement, the dissipation is positive if $w_e>w_p$, where w_e, w_p are the rates of displacement in the elastic and plastic domain. From (2.22) and (2.13) we obtain the condition for the energy dissipation to be positive in the form

$$2w_1(-x\sin\varphi_1)-\frac{k}{\sqrt{\mu_1\rho_1}\cos\varphi_1}>\frac{k(1+\pi)}{\sqrt{\rho_2\mu_2}}. \tag{2.26}$$

Thus, in the loading zone, if the profile of the incident wave does not exceed w_1° (profile 1 in Fig.2) then $D>0$, while $w_1(\Omega_1)$ is an increasing function and $D<0$ when $w_1(\Omega_1)$ begins to decrease, i.e. after passing the maximum value of the profile a plastic deformation is impossible and unloading begins. If the profile of the incident wave exceeds w_1° (profile 2 in Fig.2), then in the zone of excess on the line dividing the two media slippage (discontinuity of displacements) begins. In this case the condition for the energy dissipation to be positive will hold, while the profile of the incident wave exceeds w_1°, i.e. up to the value Ω_1^c. Henceforth $D<0$ and plastic deformation is impossible, i.e. inloading will take place.

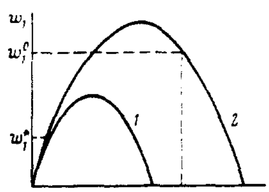

Fig.2

3. Suppose NL is the line dividing the plastic domain from the unloading zone. In the unloading zone when $x\leqslant x_N$ relations (1.12) and (1.13) hold, and they can be written in the form

$$w(x,y)=\frac{f_1(x-\sqrt{M^2-1}\,y)+f_2(x+\sqrt{M^2-1}\,y)}{2\rho_2a_2^2\sqrt{M^2-1}} \tag{3.1}$$
$$\tau_2(x,y)=\frac{f_1(x-\sqrt{M^2-1}\,y)-f_2(x+\sqrt{M^2-1}\,y)}{2c}.$$

From the boundary condition (2.5) we have when $x\leqslant x_N$

$$2w_1(-x\sin\varphi_1)=\frac{f_1(x)-f_2(x)}{2\mu_2\sqrt{M^2-1}}-\frac{\operatorname{tg}\varphi_1(f_1(x)-f_2(x))}{2\mu_1}. \tag{3.2}$$

Differentiating (3.2) and solving the equation obtained for $f_2'(x)$, we obtain

$$f_2'(x)=\frac{2R_2(x)\,g\,da_2-f_1'(x)(g-d)\sin\varphi}{\sin\varphi\,(g+d)},\quad g=\rho_1a_1\cos\varphi \tag{3.3}$$
$$R_2(x)=-\sin\varphi_1w_1'(-x\sin\varphi_1)\text{ when }x\leqslant x_N;\quad d=\rho_2a_2\cos\varphi.$$

The solution in the plastic domain (2.13) and (2.14) can be written in the form

$$w(x,y)=\frac{k(1+\pi+\varphi)-f_3(y(M-\cos\theta)(\sin\theta)^{-1}-x)}{2\rho_2a_2} \tag{3.4}$$

$$\theta(x,y)=\frac{k(1+\pi+\varphi)-f_3(y(M-\cos\theta)(\sin\theta)^{-1}-x)}{2k}. \tag{3.5}$$

From the boundary condition (2.5) we have when $x\geqslant x_N$

$$2w_1(-x\sin\varphi_1)=\frac{k(1+\pi+\varphi)-f_3(-x)}{2\rho_2a_2}-\frac{c}{\mu_1}\operatorname{tg}\varphi_1k\cos\theta. \tag{3.6}$$

Differentiating (3.5) when $y=0$ and (3.6) with respect to x and solving the set of two linear equations obtained for $f_3'(-x)$ and $\partial\theta(x)/\partial x$. we obtain

$$\frac{\partial\theta(x)}{\partial x}=-\frac{R_1(x)\rho_2a_2c}{k(g+b(\theta))},\quad f_3'(-x)=\frac{2R_1(x)\,g\rho_2a_2}{(g+b(\theta))} \tag{3.7}$$
$$R_1(x)=-2\sin\varphi_1w_1'(-x\sin\varphi_1)\text{ when }x>x_N;\quad b(\theta)=-a_2\rho_2\sin\theta.$$

Suppose $x=x_N$ is the point of the boundary, from which the propagation of the unloading wave $y=y(x)$ begins, and the velocity of the wave of the unloading when $x=x_N$ is $c^*=y'(x_N)$.

It is assumed that on the unloading wave the stresses and rates of the displacements are continuous, and in this case the following equations hold:

$$f_1\left(x + \frac{y(x)}{c_e}\right) + f_2\left(x - \frac{y(x)}{c_e}\right) = \tag{3.8}$$
$$-\frac{a_2}{c_e}\left(k(1+\pi+\varphi) - f_3\left(\frac{y(x)}{c_p} - 1\right)\right)$$
$$f_1\left(x + \frac{y(x)}{c_e}\right) - f_2\left(x - \frac{y(x)}{c_e}\right) = 2ck\cos\theta$$

where

$$c_e = -(M^2 - 1)^{-1/2}, \quad c_p = \sin\theta\,(M - \cos\theta)^{-1}$$

c_e, c_p are the velocities of the propagation of elastic and plastic waves.

Differentiating (3.8) with respect to x and writing the equations obtained for $x = x_N$ and $y(x_N) = 0$, we obtain the following set of equations for finding the initial velocity of the unloading wave c^*

$$f_1'(x_N)\left(1 + \frac{c^*}{c_e}\right) + f_2'(x_N)\left(1 - \frac{c^*}{c_e}\right) = \frac{a_2}{c_e} f_3'(-x_N)\left(\frac{c^*}{c_p} - 1\right) \tag{3.9}$$
$$f_1'(x_N)\left(1 + \frac{c^*}{c_e}\right) - f_2'(x_N)\left(1 - \frac{c^*}{c_e}\right) = -c\sin\theta f_3'(-x_N)\left(\frac{c^*}{c_p} - 1\right).$$

Using (3.3) and (3.7) to eliminate the functions $f_3'(-x_N), f_2'(x_N)$ from it, we obtain the following equations to determine c^* :

$$F(c^*) = R_1(x_N)\rho_2\sin\varphi\left(1 - \frac{c^*}{c_p}\right)\left(c\sin\theta\left(d + \frac{c^*}{c_e}g\right) + \right. \tag{3.10}$$
$$\left.\left(g + \frac{c^*}{c_e}d\right)\frac{a_2}{c_e}\right) + R_2(x_n)(g + b(\theta))\,d\left(1 - \left(\frac{c^*}{c_e}\right)^2\right) = 0.$$

It follows from (3.10) that when $R_1(x_N) = 0$, $R_2(x_N) \neq 0$ we have $c^* = c_e$, and when $R_1(x_N) \neq 0$, $R_2(x_N) = 0$ we have $c^* = c_p$. When $R_1(x_N) \neq 0$ and $R_2(x_N) \neq 0$, the sign of the quantities $F(c_p)$ and $F(c_e)$, as an analysis of Eq.(3.10) shows, depends on the sign of $R_2(x_N)$ and $R_1(x_N)$, respectively. If $R_1(x_N)$ and $R_2(x_N)$ have opposite signs, then Eq.(3.10) has at least one root which satisfies the inequality

$$|c_p| \leqslant |c^*| \leqslant |c_e|. \tag{3.11}$$

If $R_1(x_N) = R_2(x_N) = 0$, then from Eq.(3.9) we obtain $\partial\theta/\partial x = f_3'(-x_N) = 0$, and from (3.10) the quantity c^* is not determined. In this case, to determine the initial velocity of the unloading wave we shall differentiate (3.2) twice and solve the equation obtained relative to $f_2''(x)$. We have

$$f_2''(x) = \frac{2H_2(x)\,\varrho d a_2 - f_1''(x)(g - d)\sin\varphi}{\sin\varphi\,(g + d)} \tag{3.12}$$
$$H_2(x) = 2(\sin\varphi_1)^2 w_1''(-x\sin\varphi_1)\ \text{when}\ x < x_N.$$

Differentiating (3.5) twice when $y = 0$ and (3.6) with respect to x and solving the set of two linear equations obtained for $f_3''(-x)$ and $\partial^2\theta/\partial x^2$, we find

$$f_3''(-x) = -\frac{2H_1(x)\rho_2 a_2 c}{g + b(\theta)}, \qquad \frac{\partial^2\theta}{\partial x^2} = -\frac{H_1(x)\rho_2 a_2 c}{k(g - b(\theta))} \tag{3.13}$$
$$H_1(x) = 2\sin^2\varphi_1 w_1''(-x\sin\varphi_1)\ \text{when}\ x \geqslant x_N.$$

Differentiating Eq.(3.8) twice with respect to x and writing the equations obtained when $x = x_N$ and $y(x_N) = 0$, we obtain a set of equations to find the initial velocity of the unloading wave. Using (3.12) and (3.13) to eliminate the quantities $f_2''(x)$ and $f_3''(-x)$, we obtain the following equation to determine c^* :

$$F_1(c^*) = H_2(x_N)(g + b(\theta))\,d\left(1 - \left(\frac{c^*}{c_e}\right)^2\right)^2 + H_1(x_N)\sin\varphi\rho_2 \times \tag{3.14}$$
$$\left(\frac{c^*}{c_p} - 1\right)^2\left(c\sin\theta\left(d + 2\frac{c^*}{c_e}g + \left(\frac{c^*}{c_e}\right)^2 d\right) + \right.$$
$$\left.\frac{a_2}{c_e}\left(g + 2d\frac{c^*}{c_e} + \left(\frac{c^*}{c_e}\right)^2 g\right)\right) = 0.$$

At the point of the maximum $H_1(x_N) \leqslant 0$ and $H_2(x_N) \leqslant 0$. Since $F_1(c_p) < 0$, and $F_1(c_e) > 0$, then Eq.(3.14) has at least one root which satisfies the inequality (3.11). If $H_1(x_N) = H_2(x_N) = 0$, then c^* is not determined from Eq.(3.14).

Suppose all derivatives up to the n-th order of the functions $w_1^+(-x\sin\varphi_1)$ and $w_1^-(-x\sin\varphi_1)$ vanish and at least one derivative of the $n+1$-th order differs from zero. Then to determine the initial velocity of the unloading wave of Eq.(3.8) we must differentiate $n+1$ times with respect to x when $x = x_N$ and $y(x_N) = 0$, and henceforth proceed as above.

Consider finding the initial velocity of the unloading wave when there is slippage. The boundary condition (3.6) does not occur at all points of the slippage zone, and the properties of the function $w_1(-x\sin\varphi_1)$ affect the value of the initial velocity of the unloading wave when x, lying not only in the neighbourhood x_N but also in the neighbourhood x_M, are the points at which the slippage begins. In this case it is convenient to set $f_4(y(M-\cos\theta)-x\sin\theta)=f_3(y(M-\cos\theta)(\sin\theta)^{-1}-x)$ in formulas (3.4) and (3.5). Differentiating the condition of continuity of the velocities and stresses with respect to x, we obtain from (3.1), (3.4) and (3.5)

$$f_1'(x_N)\left(1+\frac{c^*}{c_e}\right)+f_2'(x_M)\left(1-\frac{c^*}{c_e}\right)= \tag{3.15}$$
$$\frac{a_2}{c_e}f_4'(0)\left(c^*(M+1)+x_N\frac{\partial\theta(x_N,0)}{\partial y}c^*\right)$$
$$f_1'(x_N)\left(1+\frac{c^*}{c_e}\right)-f_2'(x_N)\left(1-\frac{c^*}{c_e}\right)=0\,.$$

In relations (3.15), we allow for the fact that in the case of slippage when $x\in[x_N, x_M]$ the quantity θ acquires a constant value equal to π. Differentiating (3.5) with respect to y when $y=0$ and solving the equation obtained for $\partial\theta/\partial y$, we have

$$\left.\frac{\partial\theta(x_N,y)}{\partial y}\right|_{y=0}=\frac{(M+1)f_4'(0)}{2k-x_Nf_4'(0)}\,. \tag{3.16}$$

Differentiating (3.5) and (3.6) with respect to x in the neighbourhood of the point x_M, we determine

$$f_4'(0)=2k(x_M)^{-1}\,. \tag{3.17}$$

Using (3.3), (3.16) and (3.17) to eliminate the quantities $\partial\theta/\partial y$, $f_4'(0)$, $f_2'(x)$ from (3.15), we obtain the following equation to determine c^*:

$$F_2(c^*)=k\sin\varphi\,(M+1)c^*\left(g+d\frac{c^*}{c_e}\right)-\left(1-\left(\frac{c^*}{c_e}\right)^2\right)\times R_2(x_n)\,gdc_e(x_M-x_N)=0\,. \tag{3.18}$$

When there is slippage $c_p=0$. When $R_2(x_n)=0$ we have $c^*=c_p$. When $R_2(x)\neq 0$, Eq. (3.18) has at least one root which satisfies the inequality (3.11), since $F_2(0)\geqslant 0$, $F_2(c_e)\leqslant 0$. Thus, the initial velocity of the unloading wave is determined in all the cases considered. Further construction of an unloading wave can be carried out using the well-known procedure in /7/.

REFERENCES

1. BASKAKOV V.A. and BYKOVTSEV G.I., Reflection of a plane-polarized wave from a free surface in a reinforcing elastoplastic medium. PMM, Vol.35, No.1, 1971.
2. BASKAKOV V.A., Reflection of plane shear waves from a free surface in a hardening elastoplastic medium. In: Propagation of Elastic and Elastoplastic Waves. Alma-Ata: Nauka, 1973.
3. BASKAKOV V.A., Interaction of shock waves in an elastoplastic medium with reinforcement. PMTF, No.6, 1979.
4. SLEPYAN L.I., Crack mechanics. Leningrad: Sudostroenie, 1981.
5. KOVSHOV A.N., Refraction of an elastic wave into an elastoplastic half-space. Izv. AN SSSR. MTT, No.6, 1972.
6. KOVSHOV A.N., Refraction of a shear wave into the ground. Izv. AN SSSR. Fizika Zemli, No.8, 1975.
7. RAKHMATULIN KH.A. and DEM'YANOV V.A., Stability during intensive short-term loadings. Moscow: Fizmatgiz, 1961.

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